Lecture 9: Recurrence Relations

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This Lecture

1. Definition
2. Examples
3. Guess and Check
4. Binary Search
5. Characteristic Equation Method
6. The Fibonacci Sequence
7. Golden Ratio
8. Gambler's Ruin

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- A differential equation relates a function to its own derivative.
- The discrete version of a differential equation is a difference equation.
- Sometimes people call recurrence relations difference equations but really difference equations are just a type of recurrence relation.
- Some of the techniques for solving recurrence relations are almost the same as those used to solve differential equations.

Examples of Recursion Relations

Towers of Hanoi

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

Fibonacci Sequence

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

Compound Interest

Stirling Numbers

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- Guessed that the formula was $2^n - 1$
- ... and proved it with mathematical induction.

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- Given a *sorted* array of integers how might we determine if a particular integer is in the array?

Binary Search Example

```
function BinSearch(key, array, start, end)
    if end ≤ start then
        return FALSE
    else
        mid ← (end+start)/2
        if key = array[mid] then
            return TRUE
        else if key ≤ array[mid] then
            return BinSearch(key, array, start, mid)
        else if key > array[mid] then
            return BinSearch(key, array, mid, end)
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- Proof by strong induction:
- Base case $n = 2$: $\log_2(2 \cdot 2) = 1$, QED for Base Case
- Inductive Step: $\forall j < k$, $T(j) = T(\frac{j}{2}) + 1 = \log_2(j) \implies T(k) = T(\frac{k}{2}) + 1 = \log_2(k)$

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- $T(k) = T(\frac{k}{2}) + 1 = \{\log_2(k) - 1\} + 1$ by substitution.
- $T(k) = \log_2(k)$ by simplification of $-1 + 1$. QED

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- So solve for $x : \log_2(x) = \log_2(k) - 1 \implies \log_2(x) - \log_2(2) \implies \log_2(x) = \log_2(k/2)$
- $\therefore x = k/2$ Since $k/2$ is less than k we can use proof by strong induction.

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- (The term Characteristic Equation comes from Linear Algebra)
- The method works on second-order linear recurrence relations with constant coefficients. Annihilators generalize this method to any order.
- Second order recurrence: refers to two previous values of the recurrence, e.g. $T_n = T_{n-1} + T_{n-2}$

Determining the Characteristic Equation

Definition: A second-order linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form:
$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}, \exists m \in \mathbb{Z} \ni \forall k \in \mathbb{Z}, k \geq m$, where A and B are fixed real numbers with $B \neq 0$.

(The part with m and k just allows for there to be some base cases.)

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- This is true in general. A sequence of the form t^k only satisfies the 2nd order homogeneous recurrence iff it satisfies $t^2 - At - B = 0$.
- We call this equation the Characteristic Equation.

Example

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- Recall all the fun you had finding roots of quadratic formulas in Algebra I!

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- The roots are 2 and -1.
- So the only sequences of the form t^n that satisfy the recurrence are:
  $r_n = 2^0, 2^1, 2^2, 2^3, 2^4 \ldots$ and
  $s_n = (-1)^0, (-1)^1, (-1)^2, (-1)^3, (-1)^4 \ldots$

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- AND any linear combination of these sequences satisfies the recurrence: $a_n = C \cdot r_n + D \cdot s_n$, C and D can be any numbers.

Distinct Roots Theorem

All that leads us to the Distinct Roots Theorem. Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation $a_k = A \cdot a_{k-1} + B \cdot a_{k-1}$ for some real numbers A and B and k > 2.
Then $a_0, a_1, a_2, \ldots$ satisfies the closed form $a_n = C \cdot r^n + D \cdot s^n$ if the characteristic equation $t^2 - At - B = 0$ has two distinct roots r and s. Where C and D are solutions to the system of equations $a_0 = C \cdot r^0 + D \cdot s^0$ and $a_1 = C \cdot r^1 + D \cdot s^1$.
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- The Fibonacci sequence is a second-order and homogeneous with constant coefficients: A=1, B=1
- The characteristic equation is $t^2 - t - 1 = 0$
- Solving this equation for t:
  $t = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ for any quadratic polynomial:
  $at^2 + b^t + c = 0$ (Quadratic Formula).

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  $at^2 + b^t + c = 0$ (Quadratic Formula).
- $t = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$. Two distinct roots.

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- The base cases are $F_0 = 0$ and $F_1 = 1$.

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  $F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$
- Now we need to find the values of C and D using the initial conditions (differential equations), base cases (recurrences).
- The base cases are $F_0 = 0$ and $F_1 = 1$.
- So we need to solve the system of equations:
  $F_1 = 1 = C + D$
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- Solving this system gives $C = \frac{1+\sqrt{5}}{2\sqrt{5}}, D = \frac{-1+\sqrt{5}}{2\sqrt{5}}$ One way to solve this system using the method of partial fractions. If you know this method the setup is:
  $F(z) = \frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\rho_1 z} - \frac{1}{1-\rho_2 z}\right)$

Fibonacci Sequence

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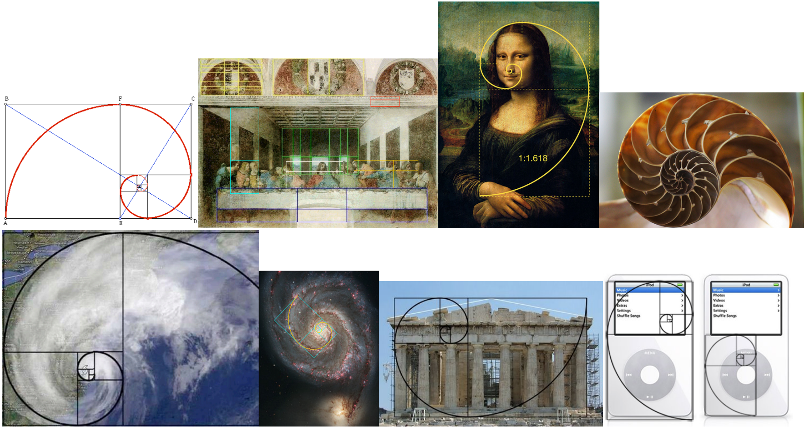
- It turns out that this value $\frac{1+\sqrt{5}}{2}$ is very special. It is called the Golden Ratio or Golden Mean and has the symbol Φ
- During the Renaissance Φ was known as the Divine Proportion

The Golden Ratio



Gambler's Ruin

Consider a game of chance. You (the player) will win \$1 or lose \$1 depending on the outcome of a coin toss. If the coin comes up heads you win if it comes up tails you lose.

You decide to play until one of two conditions are met:
1) You run out of money.
2) or you have won a target amount of money, M.

The question we would like to answer is the probability of you going bust given a starting amount of money and the target value, M.

Gambler's Ruin

- Notice that the amount of money you have in the first round is equal to your starting amount.
- Otherwise the amount of money you have depends on the amount of money you had previous round coupled with the outcome of the previous coin toss.
- So we have a sequence of values for the probability of going bust P_n given \$$n$ that depends on previous values of P_k.
- In other words we have a recurrence that we can try to define and solve.

Gambler's Ruin

- The probability of a player having \$$k$, i.e. P_k transitioning into P_{k-1} (losing a dollar) is $\frac{1}{2}$.
- The probability of transitioning to state P_{k+1} (winning a dollar) is also $\frac{1}{2}$.
- These outcomes are related by xor so we can use the addition rule of discrete probability.
- Therefore $P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1}$.

Gambler's Ruin

- Therefore $P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1}$.
- Rewriting in a form we are more used to: $0 = -2P_k + P_{k-1} + P_{k+1}$.
- Having the coefficient 2 on P_k is awkward but we can shift the sequence index by -1.
- $0 = -2P_{k-1} + P_{k-2} + P_k$.
- Rewriting: $P_k = 2P_{k-1} - P_{k-2}$.

Gambler's Ruin

- Now we have a second order homogeneous recurrence with constant coefficients.
- We know how to solve those using the Characteristic Equation method.
- What about the base cases though. Here the sequence ends under two circumstances:
- The player wins a total of $\$M$ or the player loses all their money.
- So the base cases (more like boundary conditions in this case) are $P_M = 0$ and $P_0 = 1$.
- Since the Probability of going bust when you have \$0 is 1. The Probability of going bust when you have $\$M$ is 0.

Gambler's Ruin

- Now we have everything we need to solve the recurrence.
- Characteristic Equation: $t^2 - 2t + 1 = 0$ since $A = 2$ and $B = -1$ in $P_k = AP_{k-1} + BP_{k-2}$ and the characteristic equation is $t^2 - At + B = 0$.
- Now we find the roots of the characteristic polynomial.
- Factoring: $(t-1)(t-1)$ The repeated root is $\rho = 1$.
- Using the single root theorem: $P_n = C(1)^n + nD(1)^n$.
- Solving for C and D:
  $P_0 = 1 = C + (0)D$
  $P_M = 0 = C + (M)D$
  $\therefore C = 1$
  $\therefore D = -\frac{1}{M}$
  $\therefore P_n = 1 - n\frac{1}{M}$
  $\therefore P_n = \frac{M}{M} - n\frac{1}{M} = \frac{M-n}{M}$

Gambler's Ruin

- Now we have a closed form for the recurrence and can answer the question for any target amount and starting amount of money.
- Example: What is the probability of going bust if you start with \$30 and your goal is \$120.
- Answer: $M = 120, n = 30$. $P_30 = \frac{120-30}{120} = 75\%$.
- Another Examples: $M = 500, n = 50$. $P_50 = \frac{500-50}{500} = 90\%$ chance of going bust before winning \$500.